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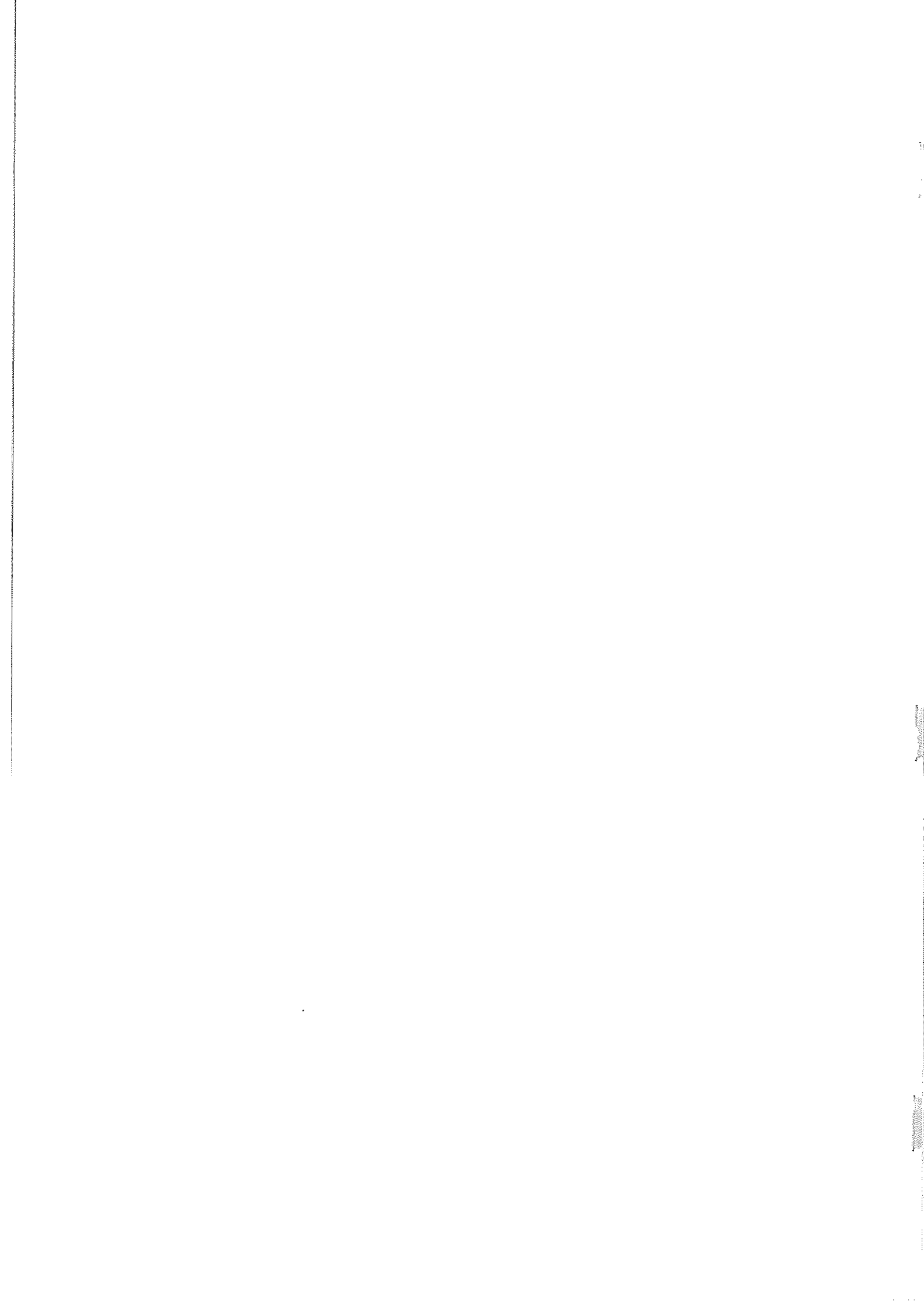
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Single-Layer Elastic Foundation—Solution
in Closed Form According to the Boussinesq
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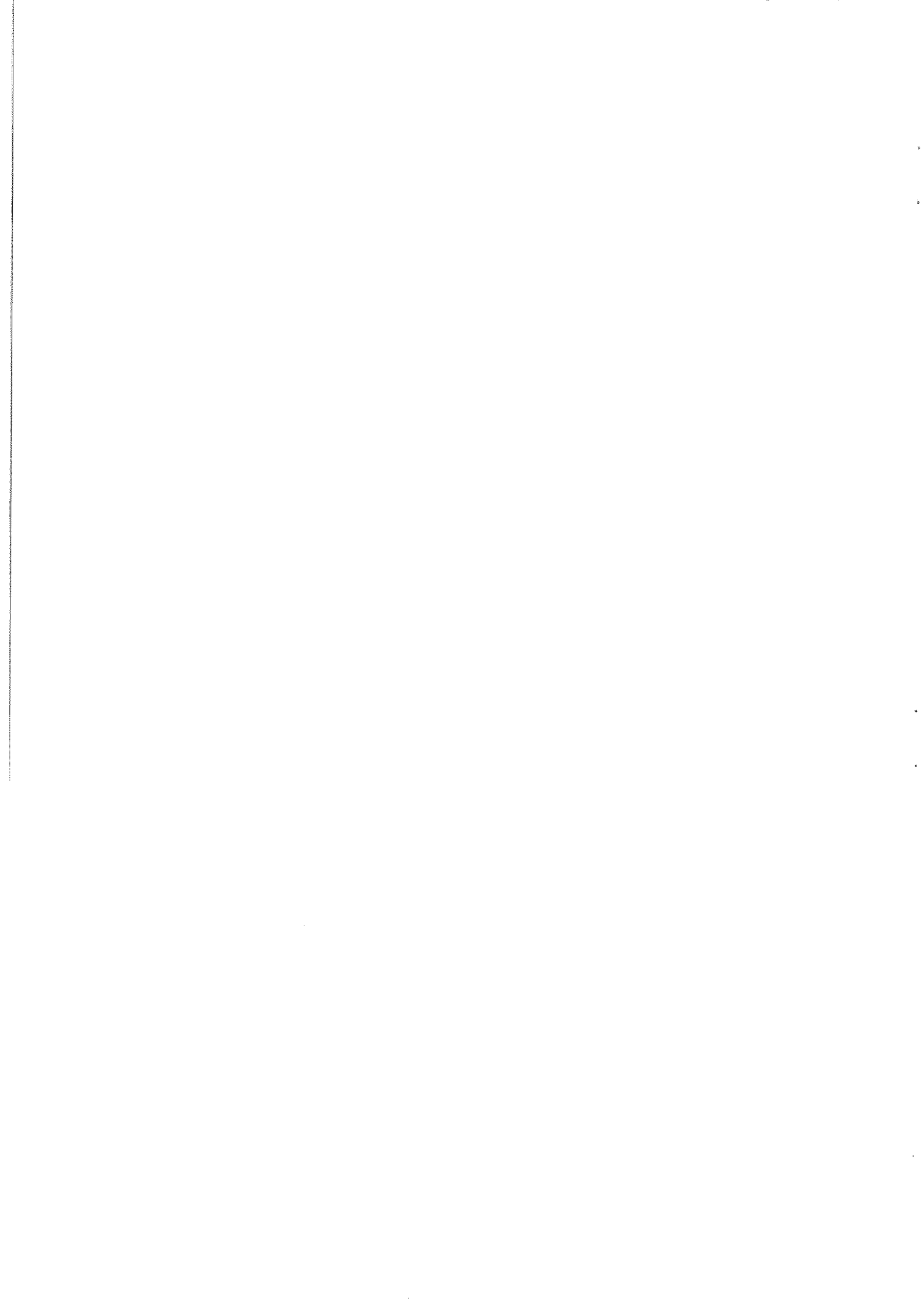
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in Closed Form According to the Boussinesq
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Bo-Göran Hellers & Olle Orrje

Presented at the International Symposium on Civil Engineering Structures Resting on Soil and Rock,
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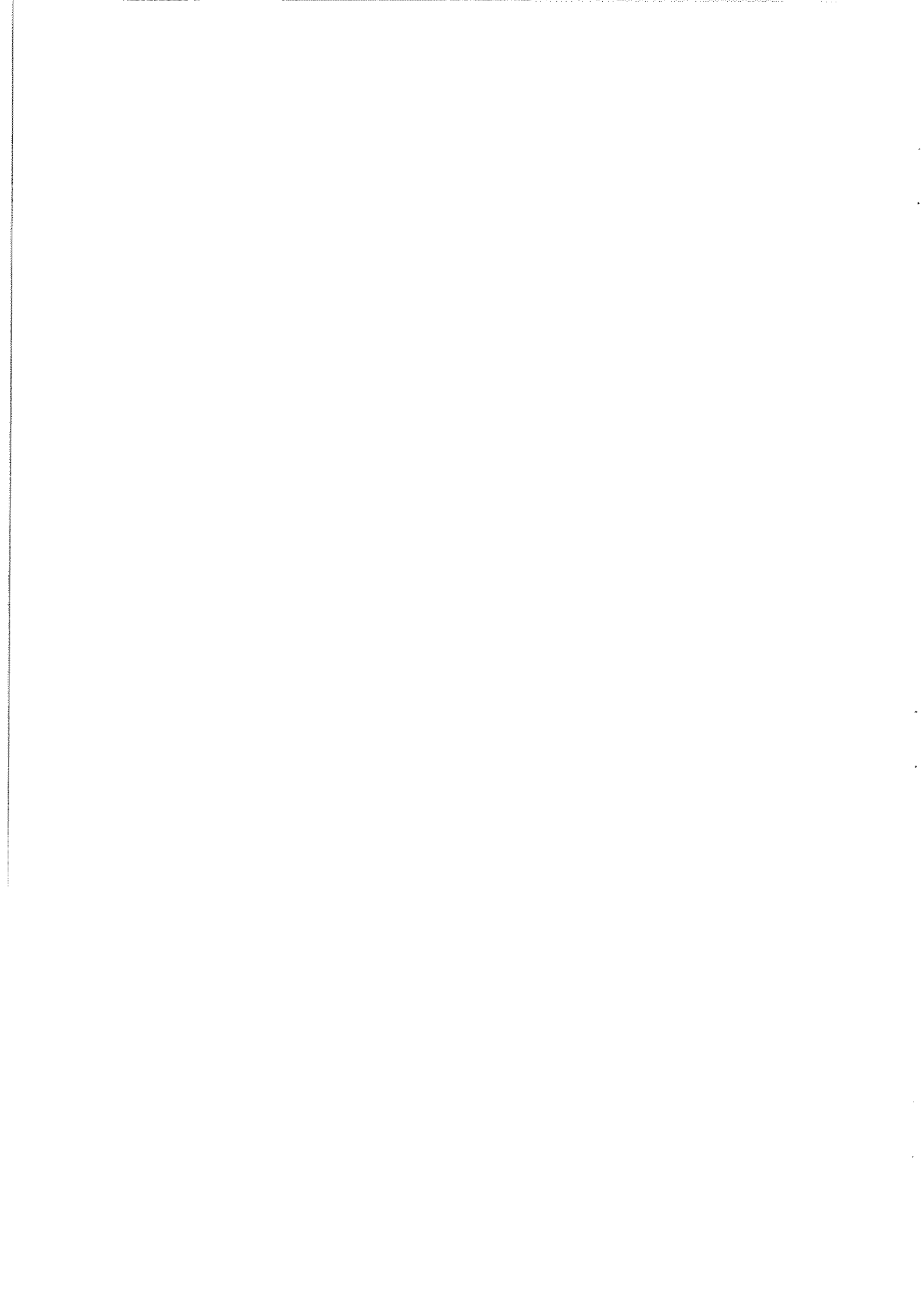


PREFACE

The research work reported in this publication was presented at the International Symposium "On Civil Engineering Structures Resting on Soil and Rocks" in Sarajevo - Jugoslavia, November 1969. The work was performed at the Royal Institute of Technology and at the Swedish Geotechnical Institute, Stockholm. It was supported by a grant from the Swedish National Council of Building Research.

Stockholm, October 1971

SWEDISH GEOTECHNICAL INSTITUTE



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Royal Institute of Technology
S t o c k h o l m
SWEDEN

CENTRICALLY LOADED INFINITE STRIP ON A
SINGLE-LAYER ELASTIC FOUNDATION - SOLU-
TION IN CLOSED FORM ACCORDING TO THE
BOUSSINESQ THEORY

S u m m a r y

1. A survey of methods to calculate the pressure distribution between an elastic beam and an elastic foundation is given. The methods are based on either the Winkler or the Boussinesq model of the subgrade reaction. In some cases the models are modified.
2. The limitations and validity of the different methods are discussed. It is stated that ordinarily the methods based on the Boussinesq model are preferable.
3. The case of a distributed load applied at the centre of an infinite strip, resting on a single layer elastic foundation, is investigated by the terms of the Boussinesq theory. The results are given in closed form and presented in a diagram giving maximum contact pressure, maximum deflection and maximum bending moment.

Introduction

The pressure distribution between an elastic beam and an elastic foundation can be calculated from two different models for the subgrade reaction.

In the first model, proposed by Winkler in 1867, the foundation is treated as equivalent to a bed of elastic springs. The contact pressure q at any point of the beam is thus assumed to be proportional to the deflection y .

In the second model proposed by Boussinesq (1885) the foundation is treated as a homogeneous isotropic elastic halfspace, characterized by the Young's modulus E and the Poisson's ratio ν . The solutions are, however, complicated and therefore seldom used in practice.

A survey of methods based on the two models is presented in this paper, and the development, limitations and validity of the methods are also discussed.

Considering elastic properties of the beam and the subgrade reaction, the Boussinesq model is closer to the true conditions of the foundation than the Winkler model. The reason that it has not been frequently used is that it takes so much labour to solve even a simple case. Thus, there is a need for solutions in closed form for some standard cases of load application. Such a solution is given in this paper for a case where a distributed load is applied at the centre of an infinite strip, resting on a single layer of elastic foundation. The solution is used in an example of practical design.

1. Calculation of contact pressures between an elastic beam and an elastic foundation by the Winkler model

The contact pressure between an elastic beam and an elastic foundation is commonly calculated from the model suggested by Winkler in 1867. This model, shown in Fig. 1.1, consists of an infinite number of independently acting elastic springs. The function of this model is that of an ideal liquid (Archimedes' principle where $p = \gamma \cdot y$). The contact pressure q at any point of the beam is proportional to the displacement y at the same point, hence

$$\frac{q}{y} = \text{constant} = k.$$

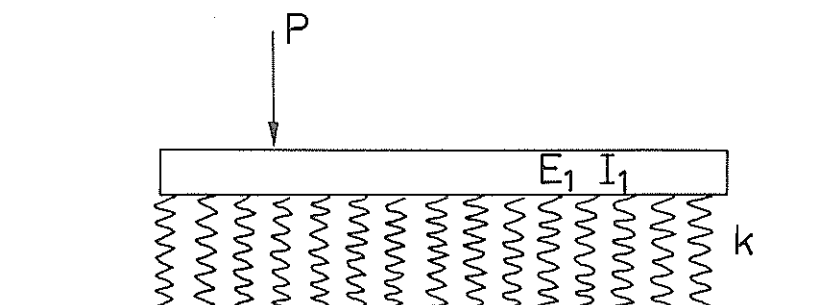


Fig. 1.1 The simple Winkler model

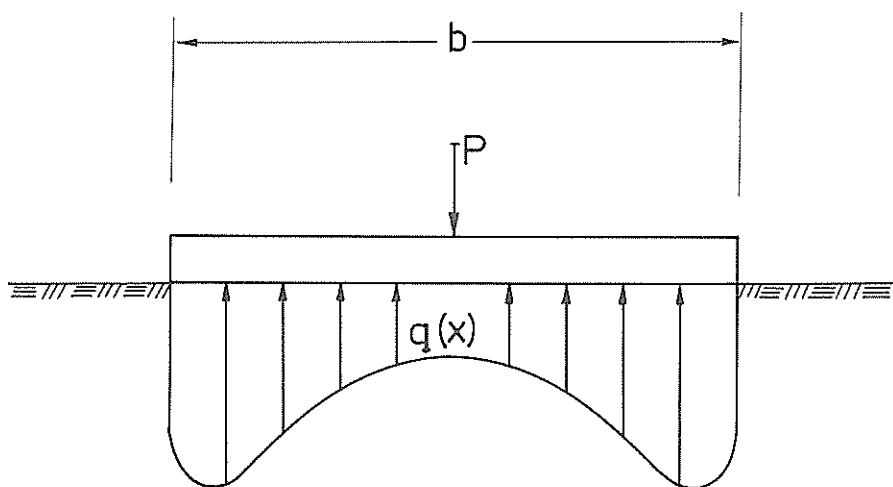


Fig. 1.2 Actual contact pressure distribution under beam resting on cohesive soil

The displacements are found from the differential equation

$$E_1 I_1 \left(\frac{d^4 y}{dx^4} \right) + k y = P(x) \quad (1.1)$$

where $E_1 I_1$ = the flexural rigidity of the beam,

y = the vertical displacement of the beam at
the section x ,

k = the constant modulus of the springs, and

$P(x)$ = the applied load on the beam at the section x .

Hetenyi (1946) has presented rigorous solutions to this equation for various end conditions. These solutions are, however, somewhat difficult to use in practice. Beside these solutions, several approximate solutions have been proposed. For example, Levinton (1947) has presented a method using redundant reactions, Gold (1948), Malter (1958) and Ray (1958) finite difference methods and Popov (1950) a semi-graphical method. Wright (1952) has proposed a method using relaxation procedures, Gazis (1958) an iterative method which is analogous with the Hardy Cross distribution method and Hendry (1958) a method based on basic function analysis.

However, the Winkler model is afflicted with a fundamental error as it does not account for any continuity of the foundation material. Consequently, the results are in many cases misleading. To improve the model, a number of modifications have been proposed, as accounted for in the following.

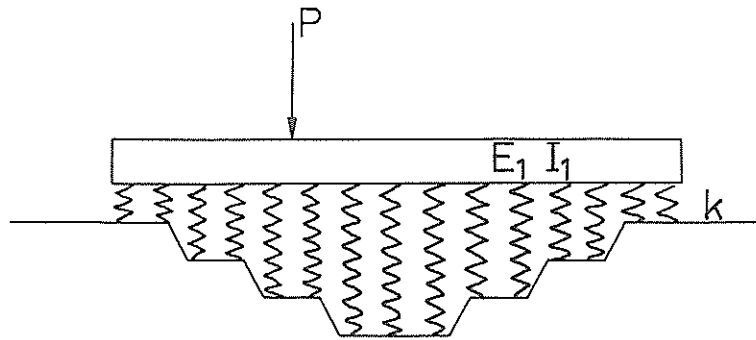


Fig. 1.3 Grashoff's modified Winkler model.
Increasing coefficient of subgrade reaction
towards the beam edges

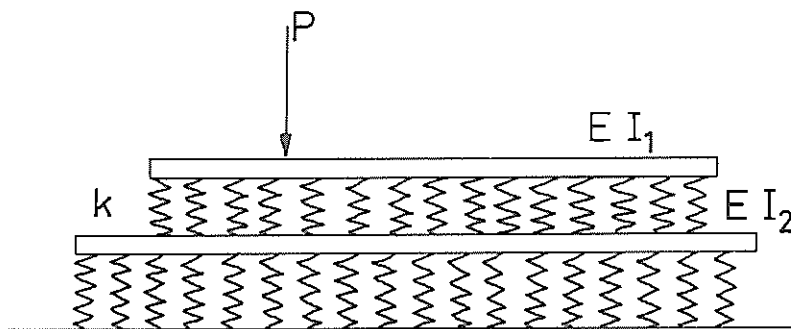


Fig. 1.4 Hetenyi's modified Winkler model.
Beam included in the foundation model

Modifications of the Winkler model

In the case of a symmetrically loaded beam with infinite flexural rigidity it can be seen from Eq. (1.1) that the simple Winkler model renders a uniform distribution of the contact pressure. From tests it is, however, well known that for many soils the contact pressures at the edge regions of a stiff beam are very high (see Fig. 1.2). Besides, according to the Boussinesq model, the pressure at the edges of a beam with infinite flexural rigidity theoretically takes infinite values. As a consequence, modifications of the Winkler model have been proposed in order that a concave stress distribution be obtained for the above case.

Graszhoff (1951) has modified the Winkler model by assuming that the coefficient k varies along the beam. As is shown in Fig. 1.3, k is assumed to increase towards the edges of the beam. Graszhoff has presented a numerical solution to this model which is a modification of the method of redundant reactions proposed by Levinton (1947).

Continuity can also be introduced by connecting the Winkler springs by strings. The forces in such strings increase with increasing curvature of the beam. The string force varies with d^2y/dx^2 . The resulting differential equation to this modified model is

$$E_1 I_1 \left(\frac{d^4 y}{dx^4} \right) - S \left(\frac{d^2 y}{dx^2} \right) + ky = P(x) \quad (1.2)$$

where S is the string constant.

As the simple Winkler bed is equivalent with an ideal liquid (Archimedes' principle $p = \gamma \cdot y$), the string connecting the Winkler springs at their upper ends is equivalent with the surface tension of a liquid.

The solution to Eq. (1.2) is similar to the corresponding basic Winkler solution with the exception that large concentrated loads are set in at the ends of the beam. These end forces may be introduced arbitrarily since the value of the surface tension σ is not generally known.

Another modified model is obtained if the springs are replaced by columns of soil. The continuity is introduced by the friction between the soil columns. It is, however, difficult to evaluate the friction since the lateral pressure varies with the degree of lateral confinement. This modified model leads to a differential equation of the form

$$E_1 I_1 \left(\frac{d^4 y}{dx^4} \right) + \alpha \left(\frac{dy}{dx} \right) + ky = P(x) \quad (1.3)$$

In order to introduce continuity into the simple Winkler model, Hetenyi (1946) has included a fictive beam in the foundation as illustrated in Fig. 1.4. With this model the following differential equation is obtained

$$A \left(\frac{d^8 y}{dx^8} \right) + B \left(\frac{d^4 y}{dx^4} \right) + Cy = P(x) \quad (1.4)$$

However, the solution to this differential equation is complicated.

The short list of modifications given here is not complete. However, all modifications have in common the fact that they cause such a complication of the theoretical problem that it is highly questionable whether the results are of such quality as to justify the calculation labor. A valuable alternative to the Winkler model which is basically different is the Boussinesq model.

2. Methods based on the stress distribution in the homogeneous isotropic elastic half-space
(Boussinesq model)

The stress distributions calculated from the modified Winkler models are dependent on the parameters k , S , γ , σ , α etc. The parameters are difficult to evaluate.

In a homogeneous isotropic elastic half-space, which contains the property of continuity, the stress distribution is correspondingly dependent on the modulus of elasticity E_s and the Poisson's ratio ν_s . The values of these parameters can be estimated for most materials.

Borowicka (1939) has presented a general solution to this problem using series expansion. But the application of the solution to practical problems leads to very extensive calculations. The solution is thus mainly of academic interest.

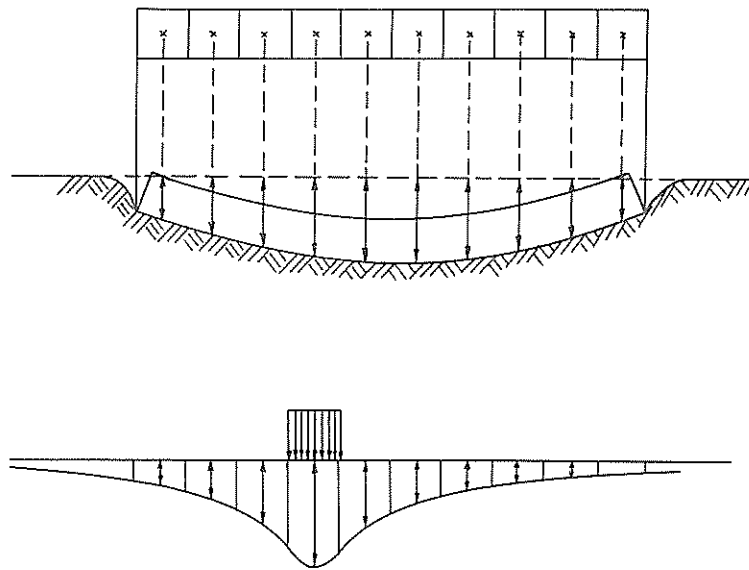


Fig. 2.1 Ohde's method. Influence diagram for the foundation deflection

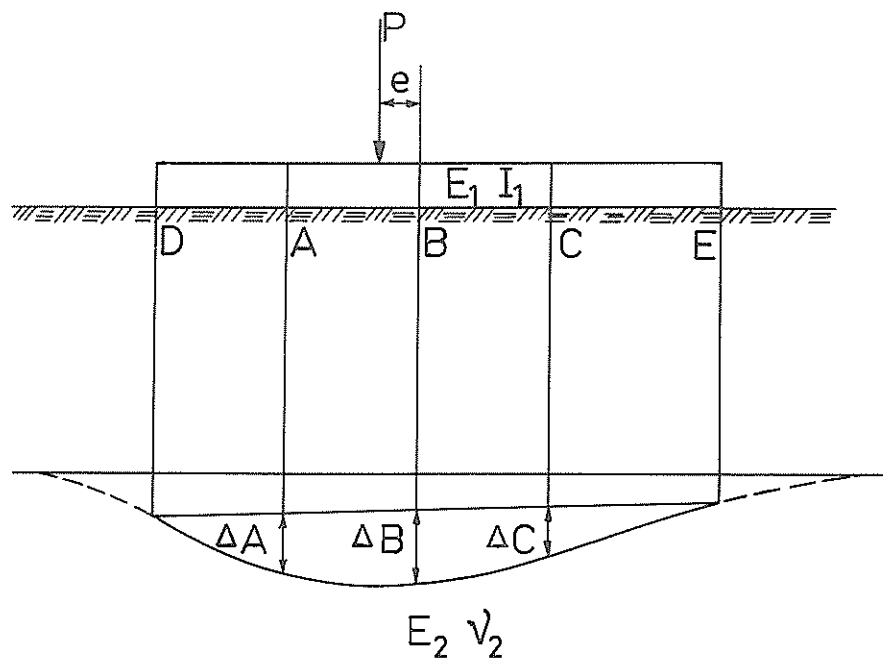


Fig. 2.2 De Beer's method. Equalizing beam and soil deflections at three points (A,B,C) along the beam

Solutions by the difference method have been given by Schleicher (1926), Habel (1938) and Ohde (1942). According to this method the beam is divided into elements of such size that the pressure distribution for each element can be assumed to be approximately uniform. Ohde's method, which is a development of the methods proposed by Schleicher and Habel, is relatively simple. The deflection of the beam is by this method calculated by superposition of the influence from the different finite elements of the beam. The corresponding deflection of the foundation is found by superposition of the influence diagram for the foundation deflection as shown in Fig. 2.1.

De Beer (1948, 1951, 1952), De Beer and Krsmanović (1951, 1952), De Beer, Lousberg and Van Beveren (1956) and De Beer and Lousberg (1964) have also studied this problem. They used the elastic half-space as a model where the modulus of elasticity E_s is constant or proportional to the applied pressure so that $E_s = C \cdot P$. This method is based on the principle that the deflection of the soil is equal to the deflection of the beam at a number of points. In this way a set of equations is obtained from which the contact pressure distribution acting along the surface between the beam and the foundation can be calculated. The contact pressure distribution has been assumed to be in the form of an n :th degree polynomial with $n + 1$ unknown coefficients.

If the center of the beam is taken as a starting-point, the pressure distribution can be written in the form

$$q(x) = q_m (a_0 + a_1x + a_2x^2 + a_3x^3 \dots + a_nx^n)$$

where q_m is the average pressure. If the pressure distribution is approximated by a fourth degree parabola, five equations are required to solve the five unknown coefficients a_0 , a_1 , a_2 , a_3 and a_4 . Three equations are obtained by equalizing beam and soil deflections at three points along the beam,

e.g. points A, B, C in Fig. 2.2. The line DE is used as a reference. Two additional equations are obtained by considering the equilibrium requirements of the beam.

Barden (1962) has proposed an approximate method for calculating the pressure distribution under beams, which is based on De Beer's method.

Krsmanović (1965) has presented diagrams of bending moments in beams subjected to concentrated loads. Krsmanović used De Beer's method, and the solution diagrams cover practically all combinations of modulus of elasticity for foundation and beam materials at different configurations of the beams.

Different modifications have been applied also to the Boussinesq model. Modified models consisting of several layers of elastic materials or elastic springs resting on an elastic foundation may be mentioned (see Rabinovitch and Herrman (1960)). Most of the investigations are analytical. Few attempts have been made to modify the theories to fit experimental results.

Important advances have recently been made by Vlasov and Leont'ev (1966). They use a modified model where the foundation consists of several elastic layers. With their method it is possible in each case to select a certain calculation scheme. The resulting equations can be solved by relatively simple mathematical means.

3. Discussion of the application of the methods

Two basic models are used for the calculation of the pressure distribution under elastic beams resting on elastic foundations: the Winkler model and a model based on the theory of elasticity where the stresses are calculated according to Boussinesq. Within each model there are several calculation methods. The most important difference between the two models is that the calculations by the Boussinesq model are based on the modulus of elasticity of the soil which is a material constant while the calculations by the Winkler model are based on a purely hypothetical quantity k , the modulus of subgrade reaction.

Terzaghi (1955) has discussed the factors affecting the value of k . Terzaghi (1955) and Vesić (1961 a) have shown that calculations give different results. However, they demonstrate that these differences can be attributed to the end conditions of the beam. For long beams the two approaches to the problem give practically the same results.

Terzaghi (1955) has for cohesive soils proposed the following relationship for the modulus of subgrade reaction of long beams

$$k = 0.67 \bar{k} \quad (3.1)$$

where \bar{k} is the modulus of subgrade reaction of a square plate with the side equal to the width of the beam.

Vesić (1961 b) has given the following relationship between the modulus k , the soil characteristics E_s , ν_s , flexural rigidity $E_b I$ and width B

$$k = 0.65 \sqrt[3]{\frac{E_s B^4}{E_b I}} \frac{E_s}{1 - \nu_s^2} \quad (3.2)$$

The fairly complicated calculations by the Winkler method are often not justified by the accuracy of the results. In fact, the results can be relatively poor, especially for short beams. However, even a mathematically exact analysis based on the theory of elasticity in the Boussinesq model does not necessarily imply that the solution is in accordance with reality, since the modulus of elasticity varies with depth and with the stress intensity (Terzaghi (1944), De Beer (1948, 1951, 1952), U.S. Waterways Experimental Station (1954)).

Furthermore, soils cannot resist the extremely high stresses which theoretically occur at the edges of a beam with high flexural rigidity. In reality, plastic zones will develop and cause a reduction and rearrangement of the stress distribution at the end zones. Besides, the methods according to the theory of elasticity do not account for the permanent deformations caused by consolidation.

Summarizing, it may be stated that the error resulting from an incorrect determination of the modulus of elasticity of the elastic foundation is generally less than the error caused by an incorrect determination of the constant k in the Winkler model. Therefore, a simple and approximate method based on an estimated modulus of elasticity of the foundation material will generally result in a more accurate stress distribution than complicated calculations based on a hypothetical modulus of subgrade reaction.

The difficulty is, however, that solutions by the Boussinesq model are generally given as parameter charts or diagrams which are troublesome to use. For a number of standard cases of load application as for instance on the infinite strip it is possible to reach solutions in closed form, which will here be demonstrated by an example in the next section.

4. Uniformly distributed load at the centre of an infinite strip on a single-layer elastic foundation
(Solution according to the Boussinesq theory in closed form.)

Strip plates which can be treated as infinitely long (length to width ratios > 4 à 5) are commonly used.

The problem is two-dimensional. Only thin plates in contact with an underlying single layer elastic foundation are considered. The exact calculation of contact pressure for such plates, whatever model is used, is rather complicated from the mathematical point of view. By using an approximate semi-infinite space model, some of the mathematical difficulties can be removed from the problem, provided the depth of the elastic foundation is equal to or larger than the width of the contact zone under the plate. The results are presented in the form of a diagram giving maximum contact pressure, maximum deflection and maximum bending moment.

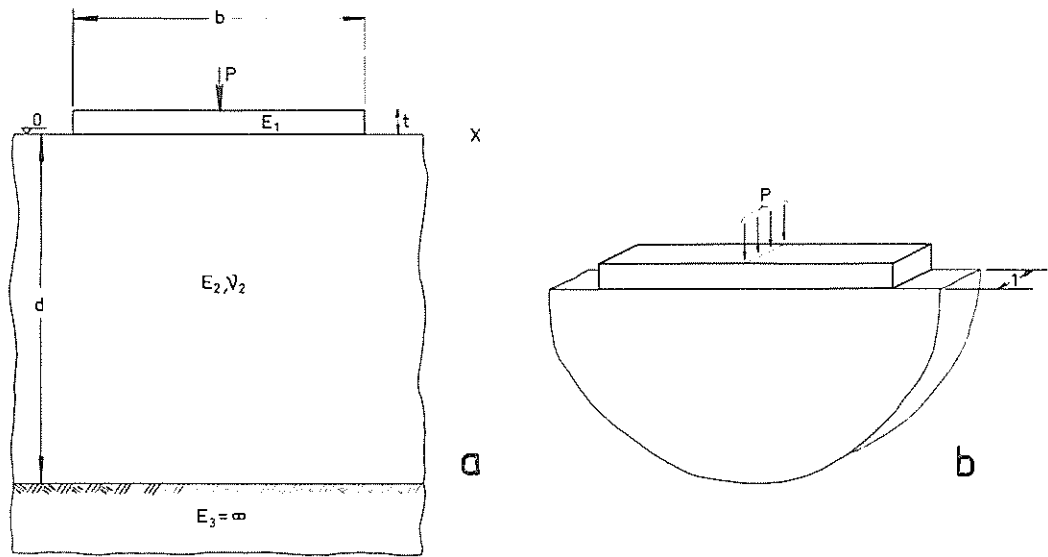


Fig. 4.1a Strip supported by a single layer elastic foundation

Fig. 4.1b Vertical disc of unity thickness supporting beam

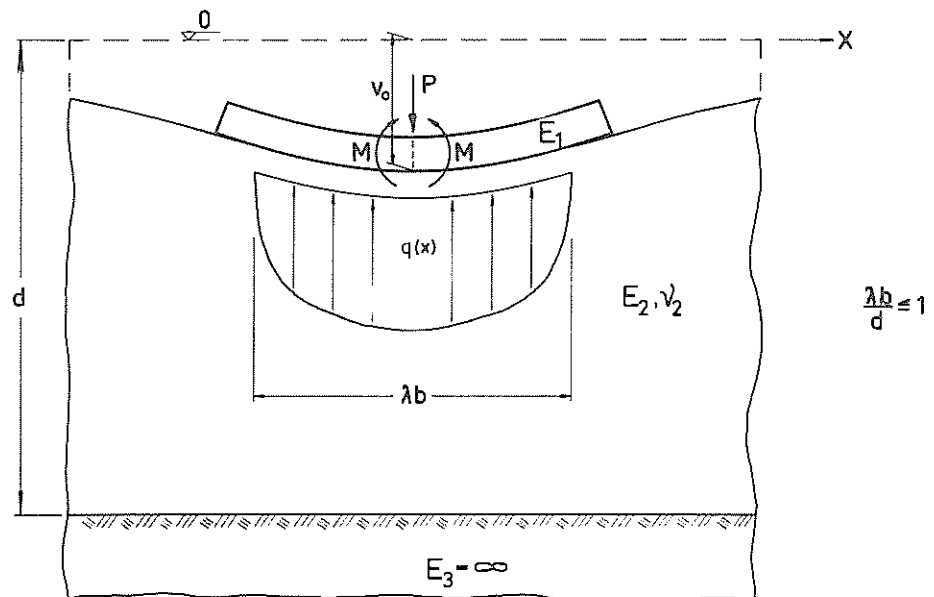


Fig. 4.2 Illustration of the deflected beam subjected to the concentrated load P

A strip supported by a single layer elastic foundation is shown in Fig. 4.1a. The uniformly distributed load P is applied at the centre of a beam. The deformations are plane. Geometrical and elastic symbols are also shown in Fig. 4.1a. Since the problem is two-dimensional it is only necessary to consider a vertical disc of unity thickness, as shown in Fig. 4.1b.

The deflected beam is illustrated in Fig. 4.2. The contact pressure $q(x)$ under the beam depends on the elastic and geometrical properties of the foundation material and of the beam. The bending moment at the centre of the beam is denoted by M and the total deflection at the centre by v_0 . The restriction that the depth of the single-layer foundation is equal to or larger than the length of the contact pressure zone, $\frac{\lambda b}{d} \leq 1$, makes it possible to obtain a solution which is independent of the depth of the single-layer foundation. Since the theory of Boussinesq is strictly valid only for an infinite foundation depth, the resulting solution is only approximate. This approximation is justified in view of the Saint-Venant's principle.

Consider a single load N acting at the end of the length λb , (Fig. 4.3). The stress conditions at the opposite end, point O , are dependent only to a limited extent on the boundary conditions at the depth d , provided that $d \geq \lambda b$. Therefore, the reaction at a point between N and O can be calculated approximately from any stress function which fulfills the condition of compatibility at each point within the single layer. The same conclusion can be drawn for the special case of a circular plate on an elastic layer with finite or infinite depth (see Vlasov and Leont'ev, 1960).

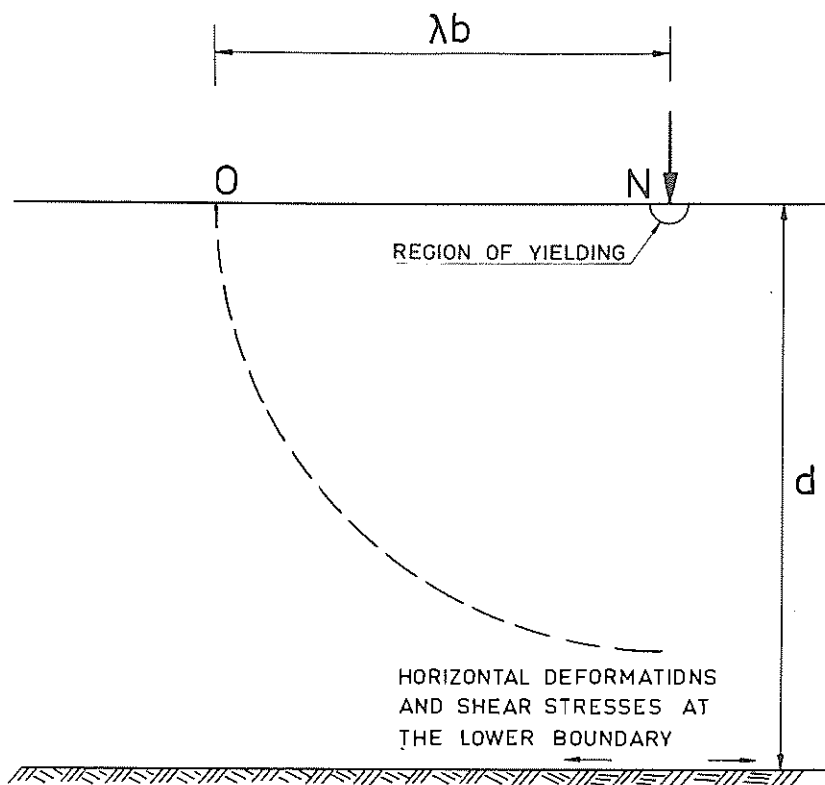


Fig. 4.3 Illustration for the application of Saint-Venant's principle.
Upper and lower boundary conditions

If the Boussinesq theory is used for a single-layer foundation, the additional assumption must be made that the actual horizontal deformations and the shear stresses along the lower boundary correspond to those from the Boussinesq theory and that the vertical deflection is zero along this boundary. Thus the application of the Boussinesq theory to the finite depth single-layer system requires an additional boundary condition at the lower boundary. Since the additional boundary condition cannot ordinarily be fulfilled, the limit $\frac{\lambda b}{d} \leq 1$ is adopted to keep the inevitable discrepancies small. The resulting error decreases as $\frac{\lambda b}{d} \rightarrow 0$.

Consider first the contact pressure $q(t)$ at the boundary of the single-layer shown in Fig. 4.4a. The resulting differential formula of the deflection according to Boussinesq (Timoshenko and Goodier, (1951)) is

$$d_t v(x) = \frac{2}{\pi E_2} q(t) \ln \left| \frac{d}{t-x} \right| dt - \frac{1 + \nu_2}{\pi E_2} q(t) dt \quad (4.1)$$

Thus

$$v(x) = \frac{2}{\pi E_2} \int_{-\lambda b/2}^{\lambda b/2} q(t) \ln \left| \frac{d}{t-x} \right| dt - \frac{1 + \nu_2}{\pi E_2} \int_{-\lambda b/2}^{\lambda b/2} q(t) dt \quad (4.2)$$

The condition of equilibrium is

$$\int_{-\lambda b/2}^{\lambda b/2} q(t) dt = P \quad (4.3)$$

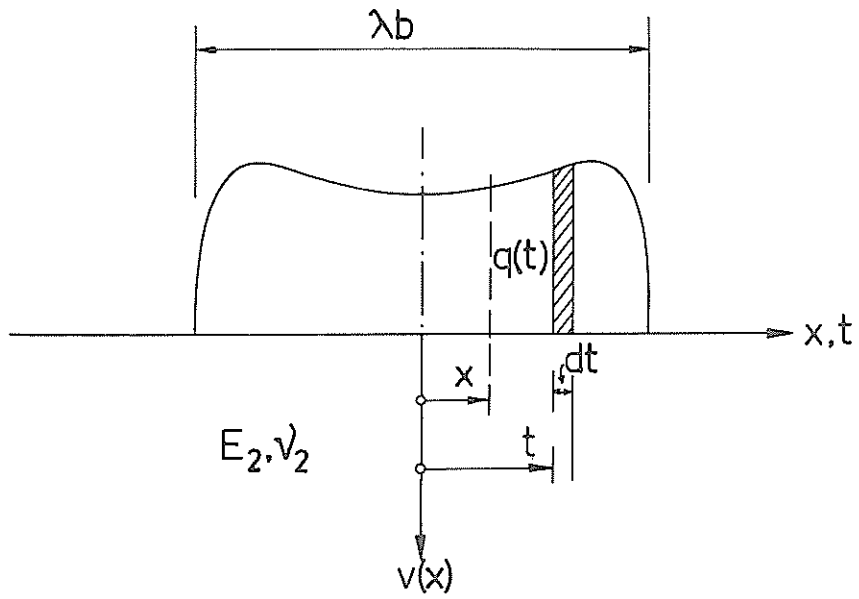


Fig. 4.4a Contact pressure at the boundary of the single layer foundation

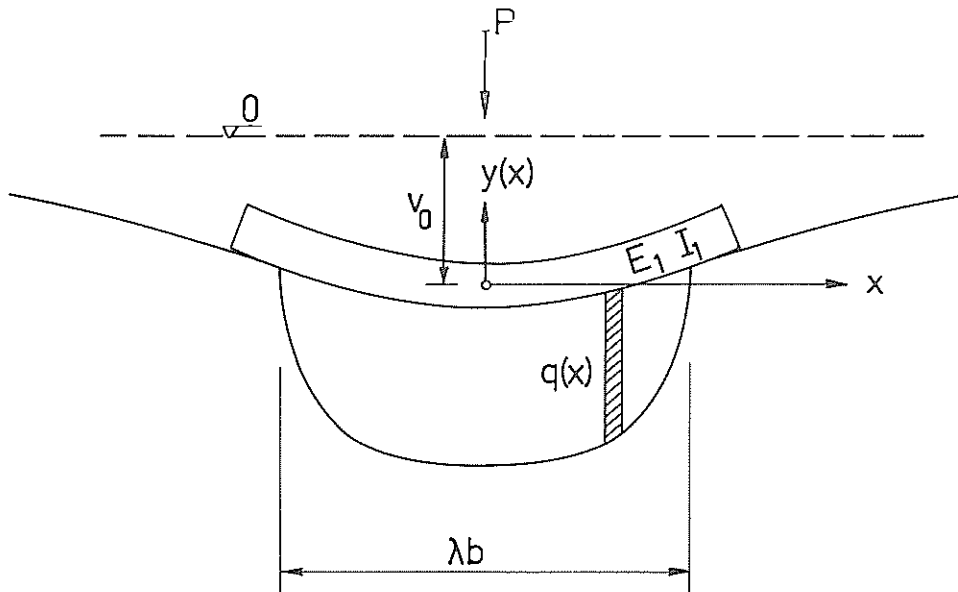


Fig. 4.4b The coordinate system for the deflection of the beam

Therefore, Eq. (4.2) may be transformed into

$$v(x) = \frac{1}{\pi E_2} \int_{-\lambda b/2}^{\lambda b/2} q(t) \ln \left(\frac{d}{t-x} \right)^2 dt - \frac{1 + \nu_2}{\pi E_2} \cdot P \quad (4.4)$$

from which the expression for the mid-section can be derived

$$v_0 = v(0) = \frac{1}{\pi E_2} \int_{-\lambda b/2}^{\lambda b/2} q(t) \ln \left(\frac{d}{t} \right)^2 dt - \frac{1 + \nu_2}{\pi E_2} \cdot P \quad (4.5)$$

Using the coordinate-system in Fig. 4.4.b the differential equation for the deflected beam can be written in the form

$$y''(x) = \frac{1}{E_1 I_1} \left[M - P \cdot \frac{x}{2} + \int_0^x q(t) \cdot (x-t) dt \right] \quad (4.6)$$

where

$$M = \int_0^{\lambda b/2} q(t) \cdot t dt \quad (4.6')$$

Integrating twice and noting that the boundary conditions are

$$y(0) = y'(0) = 0$$

the beam deflection is obtained as

$$y(x) = \frac{1}{E_1 I_1} \left[M \frac{x^2}{2} - P \frac{x^3}{12} + \int_0^x \int_0^x \left[\int_0^x q(t) (x-t) dt \right] (dx)^2 \right] \quad (4.7)$$

The condition of equal deflection of the beam and of the single-layer at each point within the interval $-\lambda b/2 \leq x \leq \lambda b/2$ leads to the identity

$$v_0 - v(x) = y(x) \tag{4.8}$$

which is the fundamental equation for the derivation of the unknown function $q(x)$.

The function must be symmetrical, i.e.

$$q(x) = q(-x) \tag{4.9}$$

The function $q(x)$ can be expanded into a power series

$$q(x) = q_m \cdot \sum_{v=0}^{\infty} a_v \left/ \frac{2x}{\lambda b} \right|^v \tag{4.10}$$

where

$$q_m = \frac{P}{\lambda b} \tag{4.10}$$

is the average pressure at the contact zone. The technique of expanding the contact pressure function into a power series has been used by De Beer and Lousberg (1964) and others. See section 2.

The condition of equal deflection, Eq. (4.8), leads to an infinite system of linear equations of the a_v :s. In approximating this system one has to confine oneself to the expansion

$$q_n(x) = q_m \cdot \sum_{v=0}^n a_v \left/ \frac{2x}{\lambda b} \right|^v \tag{4.11}$$

where n is finite.

The convergence, i.e. whether

$$\lim_{n \rightarrow \infty} q_n(x) = q(x)$$

has not been studied analytically, due to the complexity of such a task. In this connection the solution is regarded as satisfactory if, at a reasonably large n , the solution is physically possible. This has been tested by varying the stiffness of the beam. A decrease of the stiffness must result in a decrease of the contact pressure at the beam ends. For this reason it was necessary to reject the expression

$$q_n(x) = q_m \cdot \sum_{\nu=0}^n a_{\nu} \left(\frac{2x}{\lambda b}\right)^{2\nu}$$

which proved not convergent. Using the contact pressure at the ends of the beam as criterion of convergence is reasonable. The contact pressure at the ends is

$$q_n \left(\pm \frac{\lambda b}{2}\right) = q_m \cdot \sum_{\nu=0}^n a_{\nu}$$

If this expansion is to constitute the beginning of a convergent infinite series, then the following inequality must hold

$$\left| q_{n+1} \left(\pm \frac{\lambda b}{2}\right) - q_n \left(\pm \frac{\lambda b}{2}\right) \right| = q_m \cdot |a_{n+1}| < \varepsilon$$

when $n > N(\varepsilon)$

It is obvious that if

$$\sum_{\nu=0}^{\infty} a_{\nu}$$

is convergent, the expansion

$$q_n(x) = q_m \sum_{v=0}^n a_v \left(\frac{2x}{b}\right)^v$$

is an approximate solution which is valid throughout the region $|x| \leq \frac{\lambda b}{2}$, since

$$|a_j| \left|\frac{2x}{\lambda b}\right|^j \leq |a_j|$$

If it is assumed that the solution is divergent in the sense that

$$\lim_{n \rightarrow \infty} a_{n+1} \neq 0$$

$$n \rightarrow \infty$$

the divergence can easily be detected because the solution fails in physical credibility. For instance, with increasing beam stiffness the solution must ultimately result in the well-known solution for infinite stiffness. In addition the solution must vary continuously. Also the contact pressure at the beam ends must decrease to zero with decreasing beam stiffness, because the beam ends are lifted from the support when $\lambda < 1$. The divergence can also be tested by varying the number of terms in the power series expansion. A considerable discrepancy between

$$q_n \left(\pm \frac{\lambda b}{2}\right)$$

and

$$q_{n+1} \left(\pm \frac{\lambda b}{2}\right)$$

clearly indicates divergence, if n is not too small. The following expansion has been used in the computations

$$q(x) = q_m \sum_{v=0}^{11} a_v \left| \frac{2x}{\lambda b} \right|^v \quad (4.12)$$

The results are believed to be accurate enough for practical purposes.

If Eqs. (4.4) and (4.5) are used, the following expression of the relative deflection of the single-layer with regard to the centre is obtained

$$v_0 - v(x) = \frac{1}{\pi E_2} \int_{-\lambda b/2}^{\lambda b/2} q(t) \ln \left(1 - \frac{x}{t}\right)^2 dt \quad (4.13)$$

Inserting Eq. (4.11) for $q(t)$ and introducing the convenient symbol

$$\xi = \frac{2x}{\lambda b} \quad (4.14)$$

the integration involved in Eq. (4.13) is performed as follows:

$$\begin{aligned} v_0 - v(x) &= \frac{1}{\pi E_2} \int_{-\lambda b/2}^{\lambda b/2} q(t) \ln \left(1 - \frac{x}{t}\right)^2 dt = \frac{1}{\pi E_2} \left\{ \int_{-\lambda b/2}^0 q(t) \right. \\ &\ln (t - x)^2 dt - \int_{-\lambda b/2}^0 q(t) \ln t^2 dt + \int_0^{\lambda b/2} q(t) \ln (t - x)^2 dt - \\ &\left. - \int_0^{\lambda b/2} q(t) \ln t^2 dt \right\} = \frac{1}{\pi E_2} \{ I_1 - I_2 + I_3 - I_4 \} \quad (4.15) \end{aligned}$$

$$- \frac{\lambda b}{2} \leq t \leq 0$$

$$q(t) = q_m \sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v t^v$$

$$I_1 : \int_{-\lambda b/2}^0 q_m \left[\sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v t^v \right] \ln(t-x)^2 dt =$$

$$= q_m \sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v \int_{-\lambda b/2}^0 t^v \ln(t-x)^2 dt =$$

$$= q_m \sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v \left\{ \int_{-\lambda b/2}^0 \frac{t^{v+1}}{v+1} \ln(t-x)^2 - \right.$$

$$\left. - \int_{-\lambda b/2}^0 \frac{t^{v+1}}{v+1} \cdot \frac{2}{t-x} dt \right\} =$$

$$= q_m \sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v \left\{ -(-1)^{v+1} \frac{\left(\frac{\lambda b}{2}\right)^{v+1}}{v+1} \ln\left(\frac{\lambda b}{2} + x\right)^2 - \right.$$

$$\left. - \int_{-\lambda b/2}^0 \frac{2}{v+1} \left(\sum_{i=0}^v x^i t^{v-i} + \frac{x^{v+1}}{t-x} \right) dt \right\} =$$

$$= P \sum_{v=0}^n (-1)^v \frac{a_v}{v+1} \left\{ (-1)^v \left[\ln \frac{\lambda b}{2} + \ln \left| 1 + \xi \right| \right] + \right.$$

$$\left. + \sum_{i=0}^v (-1)^{v+1-i} \frac{\xi^i}{v+1-i} + \xi^{v+1} \ln \left| 1 + \frac{1}{\xi} \right| \right\}$$

$$\begin{aligned}
 I_2 &: \int_{-\lambda b/2}^0 q_m \left[\sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v t^v \right] \ln t^2 dt = \\
 &= q_m \sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v \int_{-\lambda b/2}^0 t^v \ln t^2 dt = \\
 &= q_m \sum_{v=0}^n (-1)^v a_v \left(\frac{2}{\lambda b}\right)^v \left\{ \int_{-\lambda b/2}^0 \frac{t^{v+1}}{v+1} \ln t^2 - \int_{-\lambda b/2}^0 \frac{t^{v+1}}{v+1} \cdot \frac{2}{t} dt \right\} = \\
 &= P \sum_{v=0}^n \frac{a_v}{v+1} \left[\ln \frac{\lambda b}{2} - \frac{1}{v+1} \right]
 \end{aligned}$$

$$\underline{0 \leq t \leq \frac{b}{2}}$$

$$q(t) = q_m \sum_{v=0}^n a_v \left(\frac{2}{\lambda b}\right)^v t^v$$

$$I_3: \int_0^{\lambda b/2} q_m \left[\sum_{v=0}^n a_v \left(\frac{2}{\lambda b}\right)^v t^v \right] \ln (t-x)^2 dt =$$

$$= q_m \sum_{v=0}^n a_v \left(\frac{2}{\lambda b}\right)^v \int_0^{\lambda b/2} t^v \ln (t-x)^2 dt =$$

$$= q_m \sum_{v=0}^n a_v \left(\frac{2}{\lambda b}\right)^v \left\{ \int_0^{\lambda b/2} \frac{t^{v+1}}{v+1} \ln (t-x)^2 - \int_0^{\lambda b/2} \frac{t^{v+1}}{v+1} \cdot \frac{2}{t-x} dt \right\} =$$

$$= q_m \sum_{v=0}^n a_v \left(\frac{2}{\lambda b}\right)^v \left\{ \frac{\left(\frac{\lambda b}{2}\right)^{v+1}}{v+1} \ln \left(\frac{\lambda b}{2} - x\right)^2 - \int_0^{\lambda b/2} \frac{2}{v+1} \left(\sum_{i=0}^v x^i t^{v-1} + \right. \right.$$

$$\left. + \frac{x^{v+1}}{t-x} \right) dt \right\} =$$

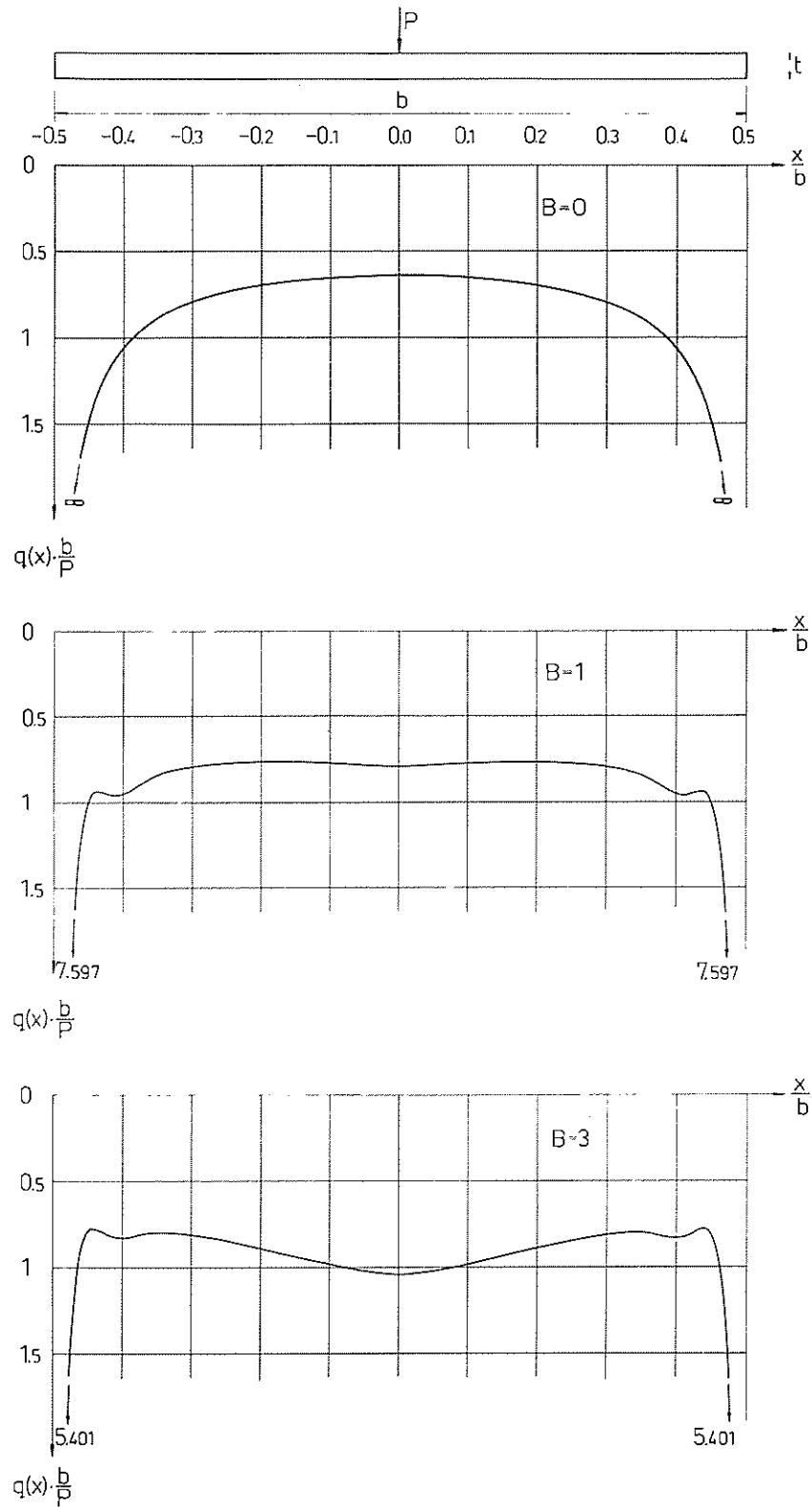


Fig. 4.5 Contact pressure distribution for increasing values of B .

$$B = \frac{E_2}{E_1} \cdot \left(\frac{b}{t}\right)^3$$

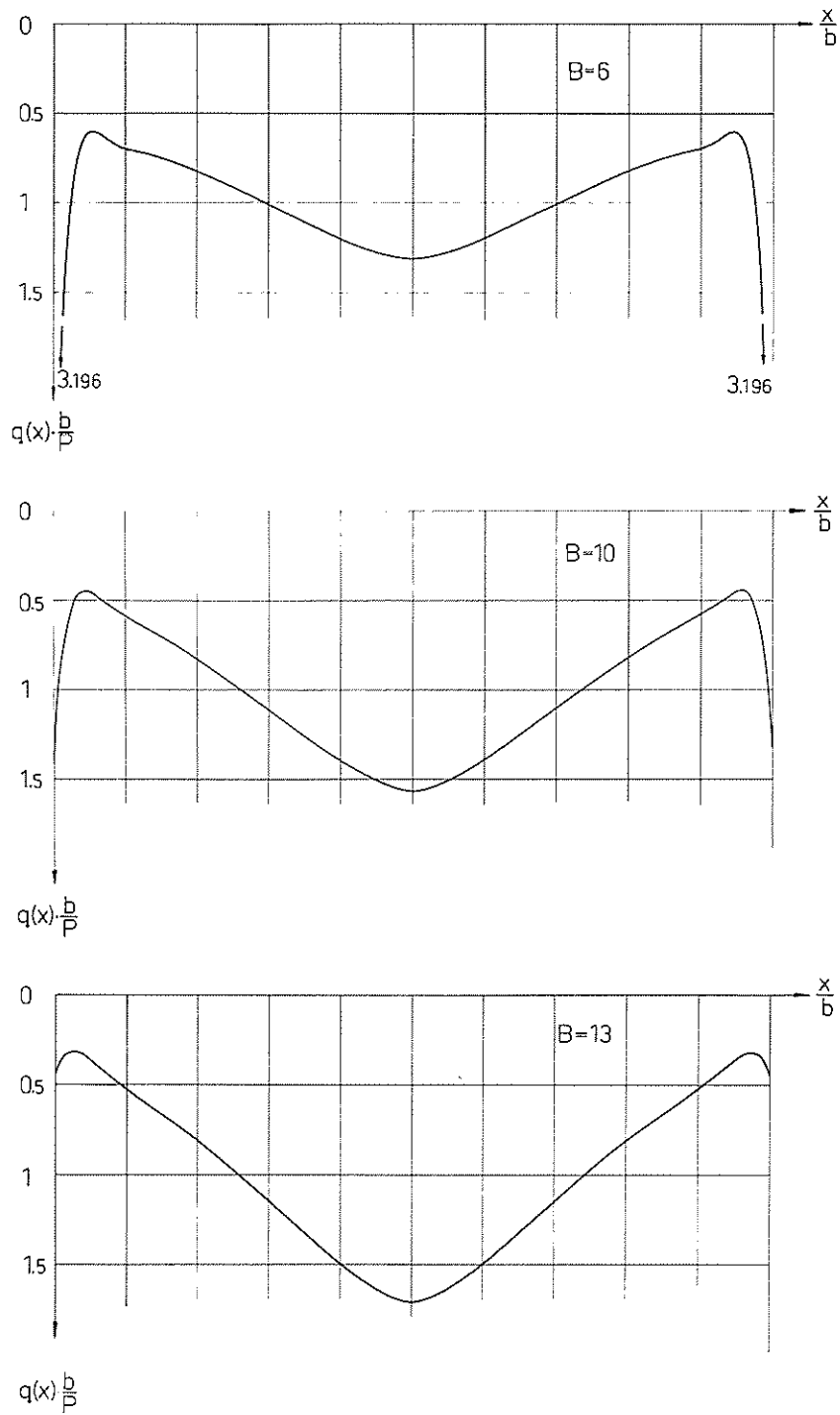


Fig. 4.5 Contact pressure distribution for increasing values of B.

$$B = \frac{E_2}{E_1} \cdot \left(\frac{b}{t}\right)^3$$

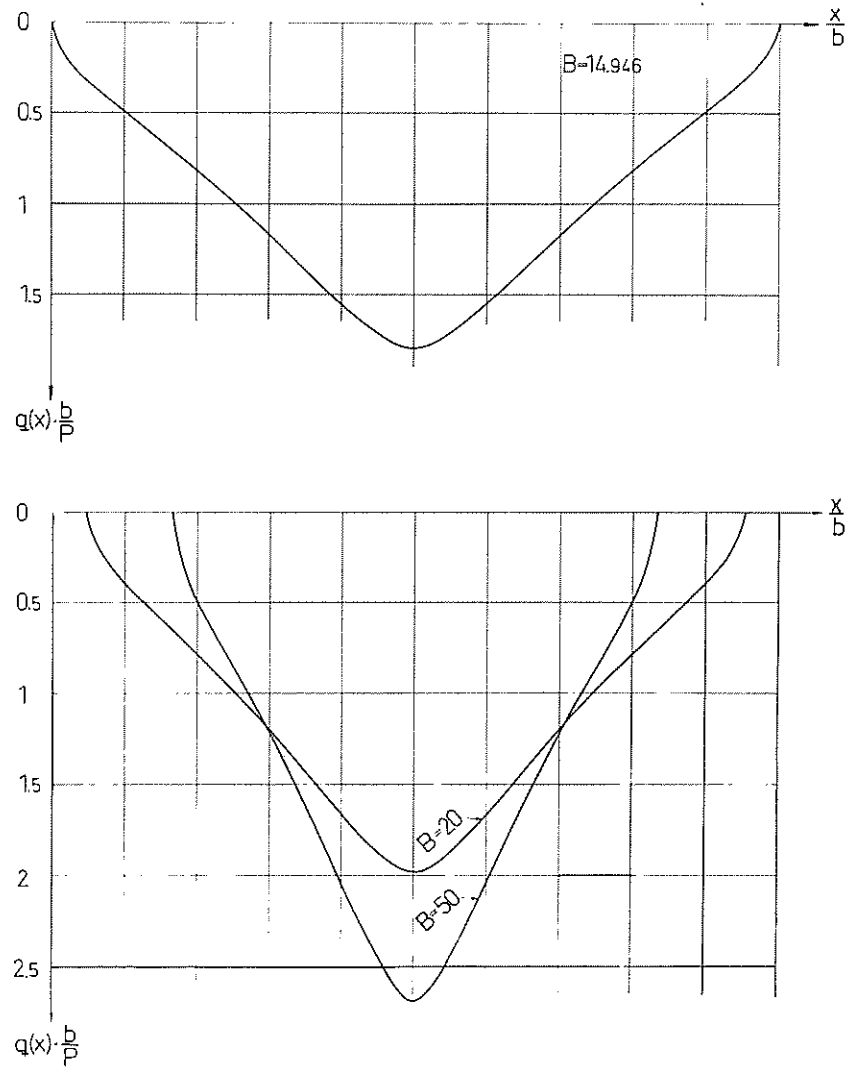


Fig. 4.5 Contact pressure distribution for increasing values of B.

$$B = \frac{E_2}{E_1} \cdot \left(\frac{b}{t}\right)^3$$

$$= P \sum_{v=0}^n \frac{a_v}{v+1} \left\{ \ln \frac{\lambda b}{2} + \ln / 1 - \xi \cdot / - \sum_{i=0}^v \frac{\xi^i}{v+1-i} - \xi^{v+1} \ln / \frac{1}{\xi} - 1 / \right\}$$

$$I_4: \int_0^{\lambda b/2} q_m \left[\sum_{v=0}^n a_v \left(\frac{2}{\lambda b} \right)^v t^v \right] \ln t^2 dt = q_m \sum_{v=0}^n a_v \left(\frac{2}{\lambda b} \right)^v \int_0^{\lambda b/2} t^v \ln t^2 dt =$$

$$= q_m \sum_{v=0}^n a_v \left(\frac{2}{\lambda b} \right)^v \left\{ \int_0^{\lambda b/2} \frac{t^{v+1}}{v+1} \ln t^2 - \int_0^{\lambda b/2} \frac{t^{v+1}}{v+1} \cdot \frac{2}{t} dt \right\} =$$

$$= P \sum_{v=0}^n \frac{a_v}{v+1} \left[\ln \frac{\lambda b}{2} - \frac{1}{v+1} \right]$$

Thus, by adding the integrals according to Eq. (4.15)

$$v_0 - v(x) = \frac{P}{\pi E_2} \sum_{v=0}^n a_v F(\xi, v) \quad (4.16)$$

where the symbol

$$F(\xi, v) = \frac{1}{v+1} \left\{ \frac{2}{v+1} + \ln(1 - \xi^2) + \sum_{i=0}^v [(-1)^{i+1} - 1] \frac{\xi^i}{v+1-i} - \right.$$

$$\left. - \xi^{v+1} \ln / \frac{1}{\xi} - 1 / + (-1)^v \xi^{v+1} \ln / 1 + \frac{1}{\xi} / \right\} \quad (4.16')$$

Similarly, by inserting the power series expansion for the contact pressure, Eq. (4.11), into Eq. (4.7) and integration, the function $y(x)$ can be obtained in terms of the coefficients a_j and the variable ξ , Eq. (4.14)

$$y(x) = \frac{P \left(\frac{\lambda b}{2} \right)^3}{E_1 I_1} \cdot \frac{\xi^2}{4} \left[\sum_{v=0}^n \left(\frac{1}{v+2} + 2 \frac{v}{v+4} \xi^{v+2} \right) a_v - \frac{\xi}{3} \right] \quad (4.17)$$

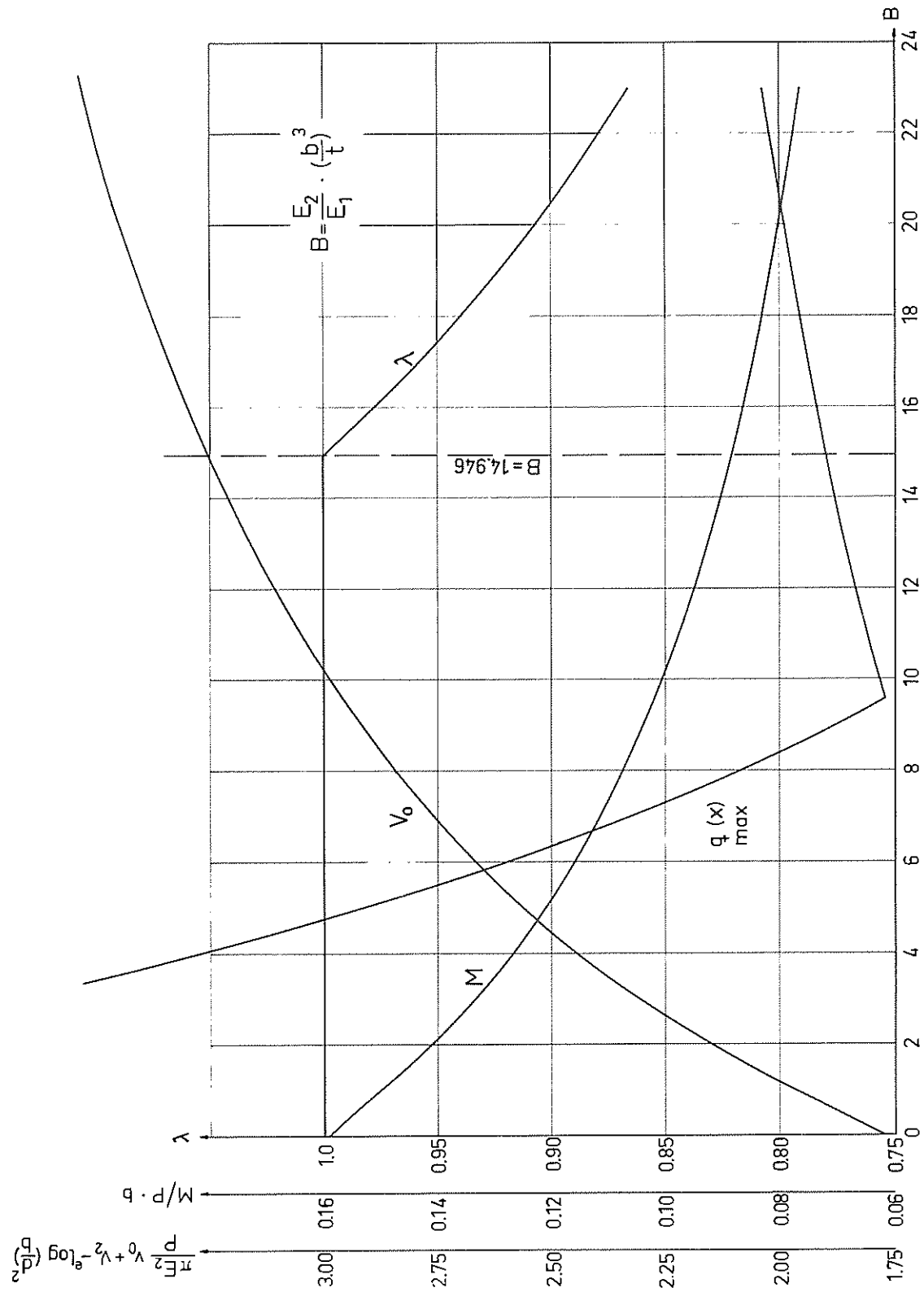


Fig. 4.6 Diagram giving maximum contact pressure, maximum deflection and maximum bending moment as function of B

For the derivation of Eq. (4.17) one has to consider the condition of equilibrium

$$\int_{-\lambda b/2}^{\lambda b/2} q(t) dt = P \quad (4.18)$$

which leads to

$$\sum_{v=0}^n \frac{a_v}{v+1} = 1$$

It is also necessary to use the expression for M Eq. (4.6') which may be transformed into

$$M = \frac{P\lambda b}{4} \sum_{v=0}^n \frac{a_v}{v+2} \quad (4.19)$$

Introducing the relative stiffness symbol

$$B = \frac{E_2}{E_1} \cdot \left(\frac{b}{t} \right)^3 \quad (4.20)$$

the condition of equal deflections Eq. (4.8) can now be written as

$$\sum_{v=0}^n \left\{ \frac{3\pi}{8} B \left[\frac{1}{v+2} + 2 \frac{v}{v+4} \xi^{v+2} \right] \cdot \xi^2 - F(\xi, v) \right\} a_v = \frac{\pi}{8} B \xi^3 \quad (4.21)$$

the symbols $F(\xi, v)$ and B defined in Eqs. (4.16') and (4.20).

For $n = 11$, Eq. (4.12), there are twelve unknown coefficients a_j ($j = 0, 1, \dots, 11$). The use of the compatibility equation, Eq. (4.21), at ten points, $\xi = 0.1, 0.2, \dots, 1.0$, results in ten equations. To these must be added the condition of equilibrium, Eq. (4.18). Furthermore, due to symmetry $q'(0) = 0$ which leads to a twelfth equation

$$a_1 = 0 \quad (4.22)$$

The resulting system can be solved conveniently with an electronic computer.

The calculated distribution of the contact pressure $q(x)$ is shown in Fig. 4.5 for different values of the relative stiffness B , Eq. (4.20). The first distribution shown in Fig. 4.5 is the well-known solution for $B = 0$. However, of special interest is the case

$$q\left(\frac{\lambda b}{2}\right) = 0$$

which marks the transition to the region characterized by the lifting of the beam ends, i.e. $\lambda < 1$.

After the coefficients a_0, \dots, a_n have been evaluated, one can calculate the bending moment M at the mid-section from Eq. (4.19), and the vertical deflection v_0 at that same section from Eq. (4.5). This equation can be transformed into

$$v_0 = \frac{P}{\pi E_2} \left\{ 2 \left(\ln \frac{2d}{\lambda b} + \sum_{v=0}^{n=11} \frac{a_v}{(v+1)^2} \right) - 1 - v_2 \right\} \quad (4.23)$$

The results are shown in Fig. 4.6. As seen, λ does not appear explicitly in the functions plotted on the vertical axis.

If these results are compared with those from the traditional Winkler-model it is interesting to notice that

$$\left(\frac{M}{Pb} \right)_{\max (B=0)} = 0.15915$$

while

$$\left(\frac{M}{Pb} \right)_{\max \text{ Winkler}} = 0.12500$$

If the bending moments M for the two models should be the same, then

$$k = 34.0 \frac{E_1 J_1}{b^4} (B - 4.28) \quad (4.24)$$

is valid throughout the interval

$$4.28 \leq B \leq 14.95 \quad (4.24')$$

Economic Design

(Minimum volume of plate)

The width b of a strip is often chosen for other reasons than those of statics. It is interesting, therefore, to investigate whether it is possible to determine an optimum plate thickness, so as to minimize the volume of the plate, given permissible stresses and deflections.

The maximum stress, σ , at the mid-section of the plate (calculated according to Navier's formula) is taken in the following as the criterion of the economic design. This stress can be calculated from the equation

$$\sigma = \frac{6M}{t^2} = 6Pb \cdot \frac{C(B)}{t^2} \quad (4.25)$$

where

$$M/Pb = C(B)$$

can be found in Fig. 4.6.

Maximize σ as the function of t in Eq. (4.25) at a constant value of b , then

$$\frac{d\sigma}{dt} = - 6Pb \cdot \frac{3BC'(B) + 2C(B)}{t^3} \Bigg|_{B = B_0} = 0 \quad (4.26)$$

Eq. (4.26) yields

$$B_0 = 14.122 \quad (4.27)$$

This solution leads to

$$t_0 = 0.4137 b \left(\frac{E_2}{E_1} \right)^{1/3} \quad (4.28)$$

and

$$\sigma_{\max} = 3.160 \frac{P}{b} \left(\frac{E_2}{E_1} \right)^{-2/3} \quad (4.29)$$

Taking $B = B_0$ in Eq. (4.27)

$$\frac{\pi E_2}{P} \cdot v_0 + v_2 - \ln \left(\frac{d}{b} \right)^2 = 3.215 \quad (4.30)$$

$$q(x)_{\max} \cdot \frac{b}{P} = 1.762 \quad (4.31)$$

The stress σ , Eq. (4.25), will decrease at a fixed value of b , if the thickness t is taken to be larger or less than t_0 .

$B_0 = 14.122$ is very close to $B = 14.946$, the boundary of the region for B in which part of the beam is not in contact with the foundation, $\lambda < 1$. To use this region in design is always uneconomic. Increasing B from B_0 is therefore of little value. This means that for practical reasons one can confine oneself to a thickness t larger than or equal to t_0 , i.e.

$$t \geq 0.414 b \left(\frac{E_2}{E_1} \right)^{1/3} \quad (4.32)$$

The lower limit of t from Eq. (4.32) is connected with the values of σ_{\max} , v_0 and $q(x)_{\max}$ from Eqs. (4.29), (4.30) and (4.31).

Example:

$$P = 40 \text{ ton/m} = 400 \text{ kp/cm}$$

$$d = 10 \text{ m}$$

$$E_1 = 170.000 \text{ kp/cm}^2$$

$$E_2 = 1000 \text{ kp/cm}^2$$

$$v_2 = 0.150$$

$$\sigma_p = 70 \text{ kp/cm}^2$$

$$v_{oP} = 1 \text{ cm}$$

$$q(x)_{\max} = 3 \text{ kp/cm}^2$$

$$\left(\frac{E_1}{E_2}\right)^{1/3} = (170)^{1/3} = 5.5397$$

Try B = 10

Largest contact pressure

$$q(x)_{\max} \cdot \frac{b}{P} = 1.568$$

$$\therefore b \geq \frac{1.568 \cdot 400}{3} = 209 \text{ cm}$$

Largest deflection

$$\frac{\pi E_2}{P} \cdot v_0 + v_2 - \ln \left(\frac{d}{B}\right)^2 = 2.987$$

$$\ln \left(\frac{d}{B}\right)^2 \leq \frac{3.142 \cdot 1000}{400} \cdot 1 + 0.150 - 2.987 = 5.018$$

$$\frac{b}{1000} \geq e^{-2.509}$$

$$b \geq 82 \text{ cm}$$

Largest stress

Take $b = 210 \text{ cm}$

$$t = 210 \cdot \frac{1}{\sqrt[3]{10}} \cdot \frac{1}{5.5397} = \frac{210}{11.935} = 17.6 \text{ cm}$$

$$\sigma = \frac{6 \cdot Pb \cdot 0.100660}{(17.6)^2} = \frac{6 \cdot 400 \cdot 210 \cdot 0.100660}{(17.6)^2} = 163.8 \text{ kp/cm}^2$$

$$> \sigma_p$$

Try B = 6

Largest contact pressure

$$q(x)_{\max} \cdot \frac{b}{p} = 3.195$$

$$b \geq \frac{3.195 \cdot 400}{3} = 426 \text{ cm}$$

Largest deflection

The necessary b is decreasing from the previous case, $B = 10$.

Largest stress

Take $b = 426 \text{ cm}$

$$t = 426 \cdot \frac{1}{\sqrt[3]{6}} \cdot \frac{1}{5.5397} = \frac{426}{10.066} = 42.3 \text{ cm}$$

$$\sigma = \frac{6 \cdot Pb \cdot 0.115866}{(42.3)^2} = \frac{6 \cdot 400 \cdot 426 \cdot 0.115866}{(42.3)^2} = 66.2 \text{ kp/cm}^2$$

$$< \sigma_p$$

Acknowledgement

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Appendix I. Notation

The following symbols are used in this paper:

A, B, C, D, E	= points along the beam in De Beer model
A	= constant
$a_0 \dots a_n$	= polynomial coefficients
B	= constant
B	= width of beam in Vesic's expression, Eq. (3.2)
B	= $\frac{E_2}{E_1} \cdot \left(\frac{b}{t}\right)^3$, relative stiffness symbol
B_0	= 14.122
b	= length of beam
C	= constant
C(B)	= $\frac{M}{Pb}$, relative bending moment symbol
d	= depth of single layer foundation
E	= Young's modulus of elasticity
$E_1 I_1$	= flexural rigidity of beam
$E_b I$	= flexural rigidity of beam in Vesic's expression, Eq. (3.2)
E_s, E_2	= Young's modulus of elasticity for subgrade material
F (ξ, ν)	= function symbol defined in Eq. (4.16)
I_1, I_2, I_3, I_4	= integrals
k	= spring constant
k	= modulus of subgrade reaction for square plate

M	= bending moment at centre of beam
N	= single load
N(E)	= real positive number
n	= degree of polynomial
O	= point at foundation boundary
P	= concentrated load
P(x)	= applied load on beam
p	= pressure per unit area
q(x)	= contact pressure under beam
q _m	= average pressure under beam
q _n (x)	= approximation with an n:th degree polynomial of q(x)
S	= string constant
t	= integration variable
t	= thickness of beam
t _o	= $0.4137 b (E_2/E_1)^{1/3}$
v(x)	= deflection function for foundation
v _o	= total deflection at the centre of beam
x	= length coordinate along the beam
y(x)	= deflection of beam
y	= deflection of subgrade in an Archimedes' model
α	= constant of lateral confinement
γ	= density of an ideal liquid
ε	= real positive number
λ	= length of contact pressure zone = λb
ν	= Poisson's ratio; also used as summation symbol
ν _s , ν ₂	= Poisson's ratio for subgrade material
ξ	= $\frac{2x}{\lambda b}$
σ	= surface tension
σ	= maximum stress at mid-section

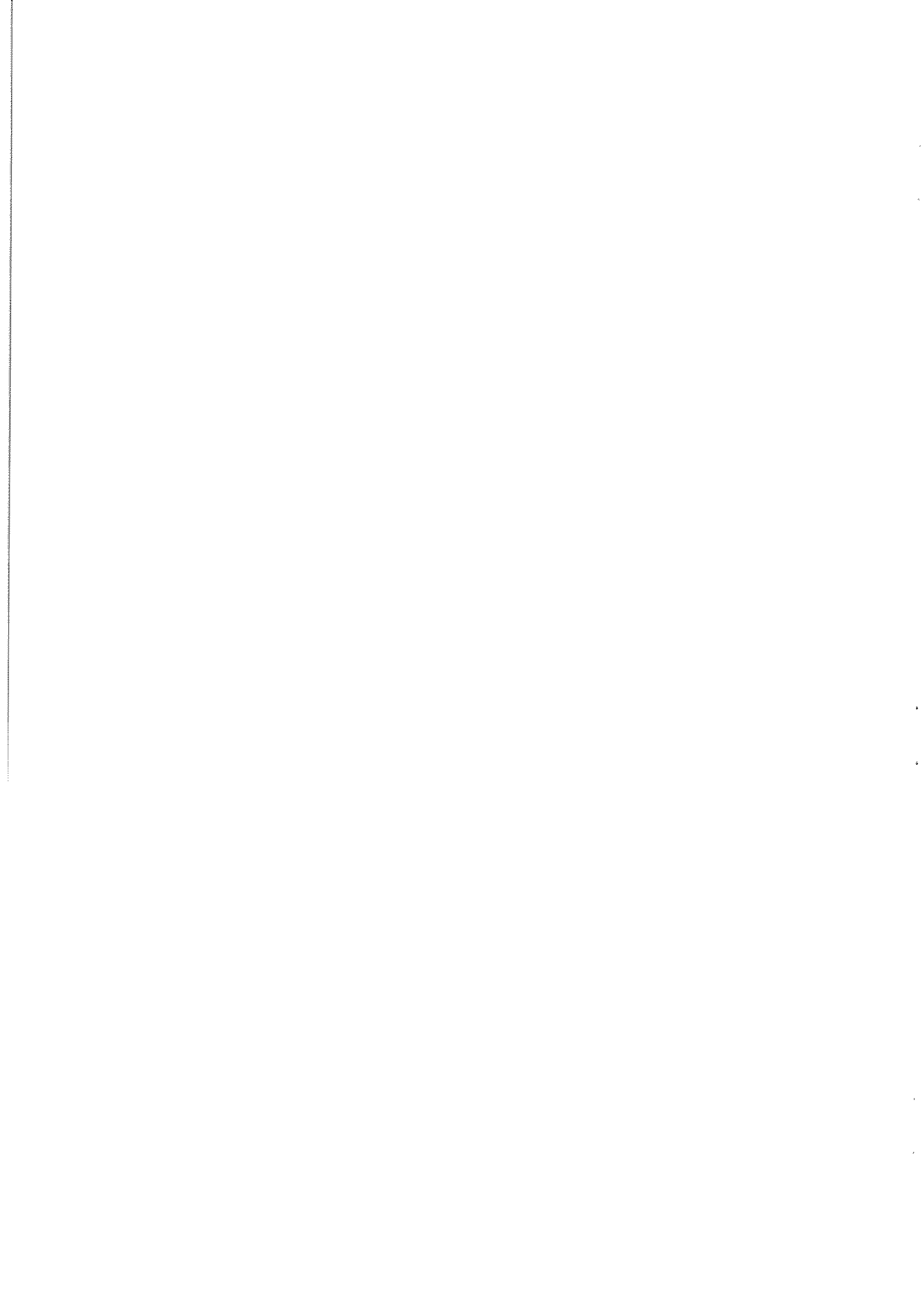
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	1. New Method of Measuring in-situ Settlements. <i>U. Bergdahl & B. Broms</i>		
	2. Effects of Pile Driving on Soil Properties. <i>O. Orrje & B. Broms</i>		
	3. End Bearing and Skin Friction Resistance of Piles. <i>B. Broms & L. Hellman</i>		
26.	Sättningar vid vägbyggnad.	1968	20:—
	Föredrag vid Nordiska Vägtekniska Förbundets konferens i Voksenåsen, Oslo 25—26 mars 1968.		
	1. Geotekniska undersökningar vid bedömning av sättningar. <i>B. Broms</i>		
	2. Teknisk-ekonomisk översikt över anläggningsmetoder för reducering av sättningar i vägar. <i>A. Ekström</i>		
	3. Sättning av verkstadsbyggnad i Stenungsund uppförd på normalkonsoliderad lera. <i>B. Broms & O. Orrje</i>		
27.	Bärförmåga hos släntberg vid statisk belastning av bergspets. Resultat från modellförsök. <i>S-E. Rehnman</i>	1968	15:—

No.		1968	Pris kr. (Sw. crs.)
28.	Bidrag till Nordiska Geoteknikermötet i Göteborg den 5-7 september 1968.	1968	15:—
	1. Nordiskt geotekniskt samarbete och nordiska geoteknikermöten. <i>N. Flodin</i>		
	2. Några resultat av belastningsförsök på lerterräng speciellt med avseende på sekundär konsolidering. <i>G. Lindskog</i>		
	3. Sättningar vid grundläggning med plattor på morän- lera i Lund. <i>S. Hansbo, H. Bennermark & U. Kihlblom</i>		
	4. Stabilitetsförbättrande spontkonstruktion för bankfyll- ningar. <i>O. Wager</i>		
	5. Grundvattenproblem i Stockholms city. <i>G. Lindskog & U. Bergdahl</i>		
	6. Aktuell svensk geoteknisk forskning. <i>B. Broms</i>		
29.	Classification of Soils with Reference to Compaction. <i>B. Broms & L. Forssblad</i>	1968	5:—
30.	Flygbildstolkning som hjälpmedel vid översiktliga grundundersökningar.	1969	10:—
	1. Flygbildstolkning för jordartsbestämning vid samhälls- planering 1-2. <i>U. Kihlblom, L. Viberg & A. Heiner</i>		
	2. Identifiering av berg och bedömning av jorddjup med hjälp av flygbilder. <i>U. Kihlblom</i>		
31.	Nordiskt sonderingsmöte i Stockholm den 5-6 oktober 1967. Föredrag och diskussioner.	1969	30:—
32.	Contributions to the 3rd Budapest Conference on Soil Mechanics and Foundation Engineering, Budapest 1968.	1969	10:—
	1. Swedish Tie-Back Systems for Sheet Pile Walls. <i>B. Broms</i>		
	2. Stability of Cohesive Soils behind Vertical Openings in Sheet Pile Walls. Analysis of a Recent Failure. <i>B. Broms & H. Bennermark</i>		
33.	Seismikdag 1969. Symposium anordnat av Svenska Geotek- niska Föreningen den 22 april 1969.	1970	20:—
34.	Något om geotekniken i Sverige samt dess roll i plane- rings- och byggprocessen. Några debattinlägg och allmänna artiklar.	1970	15:—
	<i>T. Kallstenius</i>		
	1. Geoteknikern i det specialiserade samhället. <i>B. Broms</i>		
	2. Diskussionsinlägg vid konferens om geovetenskaperna, 7 mars 1969.		
	3. Geoteknik i Sverige — utveckling och utvecklingsten- denser.		
	4. Geotekniska undersökningar och grundläggningsmeto- der.		
	5. Grundläggning på plattor — en allmän översikt.		
35.	Piles — a New Force Gauge, and Bearing Capacity Calcu- lations.	1970	10:—
	1. New Pile Force Gauge for Accurate Measurements of Pile Behavior during and Following Driving. <i>B. Fellenius & Th. Haagen</i>		
	2. Methods of Calculating the Ultimate Bearing Capa- city of Piles. A Summary. <i>B. Broms</i>		
36.	Påslagning. Materialegenskaper hos berg och betong.	1970	10:—
	1. Bergets bärförmåga vid punktbelastning. <i>S.-E. Rehnman</i>		
	2. Deformationsegenskaper hos slagna betongpålar. <i>B. Fellenius & T. Eriksson</i>		
37.	Jordtryck mot grundmurar.	1970	10:—
	1. Jordtryck mot grundmurar av Lecablock. <i>S.-E. Rehnman & B. Broms</i>		
	2. Beräkning av jordtryck mot källarväggar. <i>B. Broms</i>		
38.	Provtagningsdag 1969. Symposium anordnat av Svenska Geotekniska Föreningen den 28 oktober 1969.	1970	25:—

No.		Pris kr. (Sw. crs.)
39. Morändag 1969. Symposium anordnat av Svenska Geotekniska Föreningen den 3 december 1969.	1970	25:—
40. Stability and Strengthening of Rock Tunnels in Scandinavia. 1. Correlation of Seismic Refraction Velocities and Rock Support Requirements in Swedish Tunnels. <i>O. S. Cecil</i> 2. Problems with Swelling Clays in Norwegian Underground Constructions in Hard-Rocks. <i>R. Selmer-Olsen</i>	1971	25:—
41. Stålpålars bärförmåga. Resultat av fältförsök med lätta slagdon. <i>G. Fjellkner</i>	1971	30:—
42. Contributions to the Seventh International Conference on Soil Mechanics and Foundation Engineering, Mexico 1969.	1971	15:—
43. Centrally Loaded Infinite Strip on a Single-Layer Elastic Foundation — Solution in Closed Form According to the Boussinesq Theory. <i>B-G. Hellers & O. Orrje</i>	1972	20:—